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# HIGHLIGHTS OF TWO-PHASE CRITICAL FLOW:

## ON THE LINKS BETWEEN MAXIMUM FLOW RATES, SONIC VELOCITIES, PROPAGATION AND TRANSFER PHENOMENA IN SINGLE AND TWO-PHASE FLOWS

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**Abstract**—To improve the understanding of two-phase critical flow phenomena, both single- and two-phase flows are studied in parallel. This can be done only if compatible mathematical models are used for both flows. In particular, since the evolution of the fluid or of the mixture is, in fact, a consequence of the transfers at the wall and at the interface, it is more rational to postulate transfer laws than to assume fluid, or mixture, evolutions.

It is shown that the mathematical form of the above transfer laws is of primary importance, and it is proposed to allow for the presence, in the transfer terms, of partial derivatives of dependent variables.

The critical flow condition is discussed within the above framework. A necessary critical flow criterion is obtained by equating to zero the determinant of the set of equations describing the steady-state flow. This criterion must be complemented by the study of the compatibility conditions of the set.

It is verified that a flow is critical when disturbances, initiated downstream of some "critical" section, cannot propagate upstream of this section. A decrease of the outlet pressure has therefore no effect on the flow parameters upstream of the critical section, and the flow rate is maximum.

Examples are given to demonstrate the potentialities of the method. It is shown that appropriate assumptions on the transfer laws enable existing models to be discussed.

## 1. INTRODUCTION

The flow-rate of a compressible fluid in a pipe cannot, for given upstream conditions, be increased above some maximum, "critical" value.

In gas dynamics, this phenomenon has been studied extensively and may be considered as well known. It may occur when large expansion takes place somewhere in the fluid (engine exhaust, safety valves, restrictions, nozzles ...) and finds application in for instance, some control (sonic valves) and measurement devices.

When a two-phase fluid is compressible (e.g. gas–liquid) its flow may be subject to the same phenomenon. The latter occurs for example during the blow-down of high-pressure vessel containing a high-temperature liquid or two-phase mixture. It may also occur under less drastic conditions (see Costa & Charlety 1971).

For these reasons, the knowledge of the two-phase maximum flow-rate (for a given system under given conditions) is of paramount importance, in particular in the field of nuclear safety. As a consequence, a large number of experimental and theoretical papers on two-phase maximum flow-rates and related phenomena have been published. Among other results, previous works show the importance of slip, thermal non-equilibria, relaxation processes or, in other words, of the laws of the mass, momentum and energy transfers between the phases. An account of these works may be found in Fritte (1974) and Réocreux (1974).

However, the picture resulting from a literature search is still far from clear, and it has been felt useful to re-examine the problem, starting from what is well known (single-phase) and taking into account the specific two-phase aspects.

There are several equivalent definitions of critical conditions for single-phase flows. It cannot be *a priori* assumed that all of these definitions are also equivalent in the case of two-phase flow.

For single-phase flows, the accepted interpretation of the critical conditions may be summarized as follows:

1. The occurrence of a maximum flow-rate results from the so-called "choking"

phenomenon: a flow is choked when disturbances initiated downstream of some "critical" section cannot propagate upstream of this section. Thus a decrease of the outlet pressure, for example, has no effect on the flow parameters upstream of the critical section. As a result, the flow-rate cannot be increased further.

2. If the flow can be considered as one-dimensional, its velocity in the critical section, when choked, is equal to the propagation velocity of small pressure disturbances (velocity of sound).

3. The velocity of a disturbance depends, of course, on the nature of the thermodynamic evolution of the fluid within the disturbance. Since Laplace, the evolution in a sonic disturbance, because of its smallness, is generally supposed to be isentropic.

4. However, other kinds of evolution have been postulated. An example is the isothermal evolution (Newton).

The nature of the evolution of the fluid is usually specified *a priori* in the single phase models. A more general and rational procedure, however, is to postulate the transfer laws (cause) rather than the nature of the evolution (effect). This procedure yields models which include as particular cases all the conceivable evolutions. It is not compulsory for single phase flows, since the nature of the evolution is often known in practice. Such is not the case for two-phase flows, for which no simple assumptions can be made on the thermodynamic evolutions of the phases because of the importance, pointed out above, of the non-equilibrium processes. In this case, it is necessary to take into account the transfer laws themselves, rather than their consequences (evolution).

The purpose of this paper can now be stated explicitly: using a general mathematical model (including in particular the transfer laws), critically review the single phase flow case, and generalize to two-phase flows to improve the understanding of the two-phase critical flow phenomenon. As discussed above, the topics listed in the subtitle must be given particular consideration.

The paper is an attempt to lay the bases for a rational modeling of critical two-phase flow. Rather than presenting a new model, it examines some fundamental points about critical flow modeling, such as the consequences of the form of the transfer laws on the critical phenomenon. This results in a general mathematical form which contains most current existing models, and should enable better models to be developed.

Any comprehensive model must involve detailed transfer laws. In the present state of knowledge, this means a number of unknown coefficients which appear, from the practical point of view, as many adjustable constants: further work on the two-phase transfer laws is required. Only when this work is completed, can new models following the foregoing line be proposed. These models shall, of course, be compared to experimental data.

## 2. MATHEMATICAL MODELING

### 2.1. *General equations*

The flow of a fluid in a pipe is governed by:

1. the conservation laws: mass, momentum and energy;

2. the constitutive laws which are the mathematical model of the fluid, such as the fundamental relation and the rheological laws;

3. the boundary conditions at the pipe wall and in the inlet and outlet cross-sections.

It is beyond the scope of this paper to give the derivation of a practical set of equations from these bases, both for single- and for two-phase flow. These sets are given and discussed in Bouré *et al.* (1975).

The single-phase conservation equations used here are well-known. The two-phase conservation equations are based on works by Vernier & Delhaye (1968) and Bouré & Réocreux (1972). In spite of their nearly universal use, some of the assumptions made must be recalled here. They are classical in critical single-phase flow studies but they cannot be extended without discussion to critical two-phase flows:

1. Two-dimensional effects are neglected. Taking these effects into account would increase very much the complexity of the models and, for this reason, is not current practice. However, it must be remembered that Henry (1968) and Réocreux (1974) have shown that such effects may be present in two-phase flows, at least at low void fractions.

2. In two-phase flows, surface tension effects are neglected, and the interactions between the phases are such that the pressure is uniform in any cross-section: $p_G = p_L = p$. This assumption probably has major consequences since it affects the coupling between the phases.

3. Diffusive and turbulence effects are ignored. However, it must be remembered that the presence of many interfaces is a characteristic of two-phase flow which certainly affects both diffusion and turbulence.

Another assumption which deserves discussion concerns the form of the wall-friction and wall-heat transfer terms and, in the case of two-phase flow, of the interfacial transfer terms. These terms take into account at one and the same time some of the rheological properties of the fluids and some of the boundary conditions. The forms of all these terms are of primary importance since they affect the mathematical character of the set of equations. Most practical applications involve small gradients of the dependent variables. In these applications, it is often sufficient to assume that the transfer terms depend only on abscissa $z$, time $t$ and on the values of the other dependent variables (called hereunder $x_i$ to distinguish them, such as temperature $T$, velocity $w$ and specific entropy $s$ in the case of single-phase flow). However, as rather general flow conditions (such as high velocity, pressure or void gradients, rapid transients...) are considered in this paper, it is advisable to assume a more general dependency. For instance letting the transfer terms depend on $z$, $t$, on the $x_i$'s, and, linearly, *on the first order derivatives* of the $x_i$'s, is shown hereunder to be a significant imporvement.

The corresponding form may be regarded as a Taylor series expansion, limited to the first order. Thus, the presence of derivatives corresponds to small history and neighbourhood effects. Such effects are not surprising, at least in the case of two-phase flow, when the structure of the flow is considered, and they are probably more important than diffusive and turbulence effects.

An assumption which is similar to the foregoing one has been made by Müller (1968) for the thermodynamic theory of mixtures: this is a simple way to take into account behaviors which are less restricted than those required by the ideal gas hypothesis. The assumption is particularly well suited for processes with important non-equilibrium effects and for mixtures in which the constituents are not uniformly distributed.

An analogous situation also occurs in the field of single-phase flow turbulence and an interesting discussion on the philosophical problems raised by history and neighbourhood effects in constitutive relations may be found in Lumley (1970).

Finally, the debatable point here does not seem to be the presence of derivative in the transfer laws. It is the limitation of the Taylor expansion to the first order, and this can be justified only *a posteriori*, by comparison with experimental data.

### 2.1.1. *Single-phase flows*

The practical set of equations may be written in terms of $t$, $z$ and of the following quantities:

Main dependent variables ($x_i$): $T$, $w$, $s$.

Other dependent variables: density $\rho$, friction pressure drop per unit of length, $\mathcal{T}$, heat transfer to the flow per unit of volume $\mathcal{Q}$.

Other parameters: cross-section area $A$, projection on $z$ of the gravitational acceleration $-g\cos\theta$. With

$$A' = \frac{dA}{dz}, \quad \rho'_{T,s} = \left(\frac{\partial \rho}{\partial T}\right)_s, \quad \rho'_{s,T} = \left(\frac{\partial \rho}{\partial s}\right)_T,$$

$$a_s^2 = p'_{\rho,s} = \left(\frac{\partial p}{\partial \rho}\right)_s, \quad a_T^2 = p'_{\rho,T} = \left(\frac{\partial p}{\partial \rho}\right)_T,$$

the conservation equations are

*Mass*

$$\rho'_{T,s}\frac{\partial T}{\partial t}+\rho'_{s,T}\frac{\partial s}{\partial T}+\rho'_{T,s}w\frac{\partial T}{\partial z}+\rho\frac{\partial w}{\partial z}+\rho'_{s,T}w\frac{\partial s}{\partial z}=-\rho w\frac{A'}{A}; \qquad [1]$$

*Momentum*

$$\rho\frac{\partial w}{\partial t}+\rho'_{T,s}a_s^2\frac{\partial T}{\partial z}+\rho w\frac{\partial w}{\partial z}+\rho'_{s,T}a_T^2\frac{\partial s}{\partial z}+\mathcal{T}=-\rho g\cos\theta; \qquad [2]$$

*Energy*

$$\rho T\frac{\partial s}{\partial t}+\rho wT\frac{\partial s}{\partial z}-w\mathcal{T}-\mathcal{Q}=0. \qquad [3]$$

In the transfer laws, to avoid too large a number of unknown functions, it is further arbitrarily assumed that the involved derivatives are restricted to the derivatives $\mathrm{d}x_i/\mathrm{d}t$, with $\mathrm{d}/\mathrm{d}t=(\partial/\partial t)+w(\partial/\partial z)$.

*Friction*

$$\mathcal{T}=\tau_0-\sum_i\eta_i^*\frac{\mathrm{d}x_i}{\mathrm{d}t}=\tau_0-\eta_T\frac{\rho'_{T,s}}{w}\frac{\mathrm{d}T}{\mathrm{d}t}-\eta_w\rho\frac{\mathrm{d}w}{\mathrm{d}t}-\eta_s\frac{\rho'_{s,T}}{w}\frac{\mathrm{d}s}{\mathrm{d}t}; \qquad [4]$$

*Heat transfer*

$$\mathcal{Q}=q_0-\sum_i\zeta_i^*\frac{\mathrm{d}x_i}{\mathrm{d}t}=q_0-\zeta_T\rho'_{T,s}\frac{\mathrm{d}T}{\mathrm{d}t}-\zeta_w\rho w\frac{\mathrm{d}w}{\mathrm{d}t}-\zeta_s\rho'_{s,T}\frac{\mathrm{d}s}{\mathrm{d}t}, \qquad [5]$$

$\tau_0$, $q_0$, the $\eta$'s and the $\zeta$'s depend only on $z$, $t$ and on $T$, $w$, $s$ (the form of the coefficients in [4], [5] has been chosen to simplify further handling of the set). When all the $\eta$'s and $\zeta$'s are zero, the classical form is obtained for the transfer terms.

The set [1] to [5] is closed. Eliminating $\mathcal{T}$ and $\mathcal{Q}$ from [1] to [3] by means of [4] and [5] yields a more convenient set of three first-order partial differential equations which it is not strictly necessary to write here. Due to the assumptions made, this set is linear with respect to the derivatives, which is also the case of most existing single-phase flow models. Hence the numerical solution of the set involves the determinant $\Delta$ of the coefficients of the $\partial/\partial z$ terms, and, for transients, the determinant $\Delta_t$ of the coefficients of the $\partial/\partial t$ terms (see sections 2.2. and 2.3.).

2.1.2. *Two-phase flows*

The subscripts $K = G$ or $L$ being used for quantities relating respectively to the gas and to the liquid phase, the practical set of equations may be written in terms of $t$, $z$, and of the following quantities:

Main dependent variables ($x_i$): $\alpha_G$ (void fraction), $p$, $w_L$, $\Delta w$ (velocity difference $w_G - w_L$), $\Delta h_K$ (difference $h_K - h_{K\text{sat}}$ between the enthalpy of phase $K$ and the saturation enthalpy of the same phase)

Other dependent variables: $\rho_K$, $\mathcal{T}_K$, $\mathcal{Q}_K$ and $M$, $(MV)$, $(MH)$, (respectively mass-, momentum-, and energy-transfer terms from the liquid to the gas-phase per unit of volume, the complete expression of which can be found in Bouré *et al.* 1975). With

$$\rho'_{K/p,\Delta} = \left(\frac{\partial \rho_K}{\partial p}\right)_{\Delta h_K}, \quad \rho'_{K/h,p} = \left(\frac{\partial \rho_K}{\partial h_K}\right)_p, \quad h'_{K\mathrm{sat}} = \frac{\mathrm{d}h_{K\mathrm{sat}}}{\mathrm{d}p}, \quad \alpha_L = 1 - \alpha_G,$$

the conservation equations are (the upper sign is valid for $K = G$ and the lower sign is valid for $K = L$)

*Mass*

$$\pm \rho_K \frac{\partial \alpha_G}{\partial t} + \alpha_K \rho'_{K/p,\Delta} \frac{\partial p}{\partial t} + \alpha_K \rho'_{K/h,p} \frac{\partial \Delta h_K}{\partial t} \pm \rho_K w_K \frac{\partial \alpha_G}{\partial z} + \alpha_K w_K \rho'_{K/p,\Delta} \frac{\partial p}{\partial z}$$
$$+ \alpha_K \rho_K \frac{\partial w_L}{\partial z} + \left(\alpha_G \rho_G \frac{\partial \Delta w}{\partial z}\right)_{\text{if } K=G \text{ only}} + \alpha_K w_K \rho'_{K/h,p} \frac{\partial \Delta h_K}{\partial z} \mp M = -\alpha_K \rho_K w_K \frac{A'}{A}; \qquad [6_K]$$

*Momentum*

$$\pm \rho_K w_K \frac{\partial \alpha_G}{\partial t} + \alpha_K w_K \rho'_{K/p,\Delta} \frac{\partial p}{\partial t} + \alpha_K \rho_K \frac{\partial w_L}{\partial t} + \left(\alpha_G \rho_G \frac{\partial \Delta w}{\partial t}\right)_{\text{if } K=G \text{ only}} + \alpha_K w_K \rho'_{K/h,p} \frac{\partial \Delta h_K}{\partial t}$$
$$\pm \rho_K w_K^2 \frac{\partial \alpha_G}{\partial z} + \alpha_K (1 + w_K^2 \rho'_{K/p,\Delta}) \frac{\partial p}{\partial z} + 2\alpha_K \rho_K w_K \frac{\partial w_L}{\partial z} + \left(2\alpha_G \rho_G w_G \frac{\partial \Delta w}{\partial z}\right)_{\text{if } K=G \text{ only}}$$
$$+ \alpha_K w_K^2 \rho'_{K/h,p} \frac{\partial \Delta h_K}{\partial z} \mp (MV) + \mathcal{T}_K = -\alpha_K \rho_K g \cos\theta - \alpha_K \rho_K w_K^2 \frac{A'}{A}; \qquad [7_K]$$

*Energy*

$$\pm \rho_K \left(h_K + \frac{w_K^2}{2}\right) \frac{\partial \alpha_G}{\partial t} + \alpha_K \left[\left(h_K + \frac{w_K^2}{2}\right) \rho'_{K/p,\Delta} + \rho_K h'_{K\mathrm{sat}} - 1\right] \frac{\partial p}{\partial t} + \alpha_K \rho_K w_K \frac{\partial w_L}{\partial t}$$
$$+ \left(\alpha_G \rho_G w_G \frac{\partial \Delta w}{\partial t}\right)_{\text{if } K=G \text{ only}} + \alpha_K \left[\left(h_K + \frac{w_K^2}{2}\right) \rho'_{K/h,p} + \rho_K\right] \frac{\partial \Delta h_K}{\partial t} \pm \rho_K w_K \left(h_K + \frac{w_K^2}{2}\right) \frac{\partial \alpha_G}{\partial z}$$
$$+ \alpha_K w_K \left[\left(h_K + \frac{w_K^2}{2}\right) \rho'_{K/p,\Delta} + \rho_K h'_{K\mathrm{sat}}\right] \frac{\partial p}{\partial z} + \alpha_K \rho_K \left(h_K + \frac{3 w_K^2}{2}\right) \frac{\partial w_L}{\partial z}$$
$$+ \left[\alpha_G \rho_G \left(h_G + \frac{3 w_G^2}{2}\right) \frac{\partial \Delta w}{\partial z}\right]_{\text{if } K=G \text{ only}} + \alpha_K w_K \left[\left(h_K + \frac{w_K^2}{2}\right) \rho'_{K/h,p} + \rho_K\right] \frac{\partial \Delta h_K}{\partial z}$$
$$\mp (MH) - \mathcal{Q}_K = -\alpha_K \rho_K w_K g \cos\theta - \alpha_K \rho_K w_K \left(h_K + \frac{w_K^2}{2}\right) \frac{A'}{A}. \qquad [8_K]$$

The functions $\mathcal{T}_K$ and $\mathcal{Q}_K$ have the same form as in the case of single-phase flow. However, for the sake of simplicity, only the particular forms

$$\mathcal{T}_K = \tau_{K,0}, \qquad [9_K]$$

$$\mathcal{Q}_K = q_{K,0} \qquad [10_K]$$

are used in this paper.

The functions $M$, $(MV)$, $(MH)$ are

$$M = M_0 - \sum_i \lambda^t_{x_i} \frac{\partial x_i}{\partial t} - \sum_i \lambda^z_{x_i} \frac{\partial x_i}{\partial z}, \qquad [11]$$

$$(MV) = (MV)_0 - \sum_i \mu^t_{x_i} \frac{\partial x_i}{\partial t} - \sum_i \mu^z_{x_i} \frac{\partial x_i}{\partial z}, \qquad [12]$$

$$(MH) = (MH)_0 - \sum_i \nu^t_{x_i} \frac{\partial x_i}{\partial t} - \sum_i \nu^z_{x_i} \frac{\partial x_i}{\partial z}, \qquad [13]$$

$\tau_{K,0}$, $q_{K,0}$, $M_0$, $(MV)_0$, $(MH)_0$, the $\lambda$'s, the $\mu$'s and the $\nu$'s depend only on $z$, $t$ and on the $x_i$'s.

The set [$6_K$]–[$10_K$], [11]–[13] is closed. Eliminating $\mathcal{T}_K$, $\mathcal{Q}_K$, $M$, $(MV)$, $(MH)$ from [$6_K$]–[$8_K$] by means of [$9_K$]–[13] yields a more convenient set of six first-order partial differential equations which it is not necessary to write here. As in the case of single-phase flow, this set is linear with respect to the derivatives of the $x_i$'s, which is also the case for most existing two-phase flow models. As in the case of single-phase flow, this leads to the consideration of the determinant $\Delta$ of the coefficients of the $\partial/\partial z$ terms, and, for transients, of the determinant $\Delta_t$ of the coefficients of the $\partial/\partial t$ terms.

This important feature, which is shared by most practical single- and two-phase flow models, is the basis of the following discussion.

### 2.2. *Critical flow condition*

Several approaches have been proposed to obtain a critical flow criterion. Some of them are based on propagation phenomena; others involve the vanishing condition of some partial derivative of the flow-rate. In any case, as pointed out by Katto & Sudo (1973), the critical flow condition must result from the mathematical model of the system. A practical viewpoint is adopted here: starting from a set of inlet values for the $x_i$'s the steady state values of all the dependent variables are calculated while proceeding downstream along the pipe, using the steady state version of either the single-phase or the two-phase set of equations. This process is not claimed to be the best to study critical flow. However, the results it yields apply to any flow model which involves only first order partial differential equations, linear with respect to the derivatives, which again is a very current case. The process is schematized by the flow-chart of figure 1, which is a support for the following discussion. Usually, not all the $x_i$'s are given at the inlet: the corresponding values are guessed and subsequently adjusted by iteration.

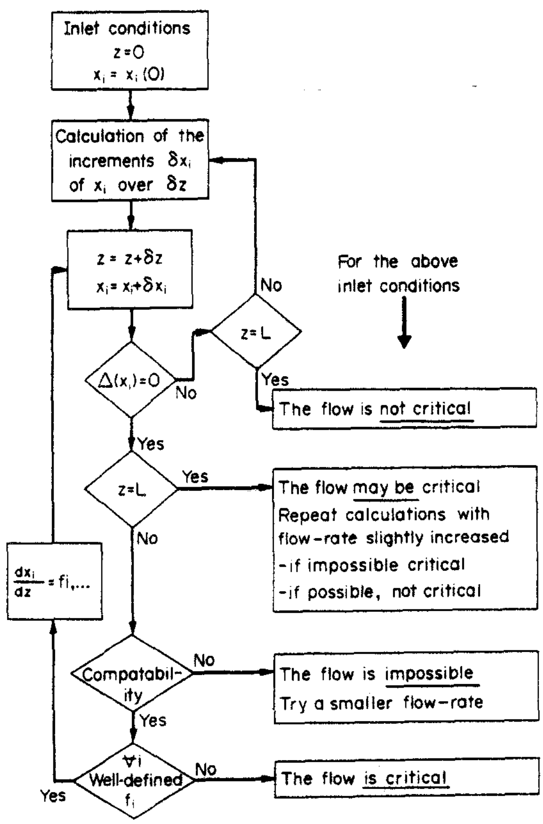


Figure 1.

In a given cross-section, the calculation of the first order derivatives of the $x_i$'s with respect to $z$ involves a linear algebraic set of equations whose determinant is $\Delta$ (see sections 2.11 and 2.12). If higher order derivatives of the $x_i$'s are needed, they can be calculated by differentiating the original set of equations; the new sets have the same determinant $\Delta$.

As long as $\Delta \neq 0$, no singularity appears in the process, and complete numerical solution is obtained, the $x_i$'s being obtained as Taylor expansions with respect to $z$. If $\Delta \neq 0$ up to the end of the channel ($z = L$), there is one and only one solution; the inlet conditions can be slightly modified; the flow is not critical.

If $\Delta$ vanishes in any cross-section, the problem is either *impossible* or *indeterminate*, depending on the fulfillment of a certain compatibility condition. In order to avoid lengthy developments, some particular cases are excluded here *a priori*, namely: $w = 0$ (single-phase flow), $w_K = 0$, $\alpha_K = 0$, $\alpha_K = 1$ (two-phase flow). It is also assumed that the inlet conditions are such that the determinant $\Delta$ of the set of equations does not vanish at the inlet of the pipe. Thus, the calculation of the steady-state values of the dependent variables can be started inasmuch as these inlet conditions are known.

$N_i$ being the determinant $\Delta$ where the $i$th column has been replaced by the R.H.S. members of the equations of the set, the compatibility condition, which yields a single equation, is that all the determinants $N_i$ vanish simultaneously when $\Delta = 0$. ($N_i$ stands here, as $x_i$ for the plural $N_i$'s). Hence, the gradients $N_i/\Delta$ of the dependent variables $x_i$ take the indeterminate form 0/0 in the cross-section under consideration, and are not in general infinite, as often assumed. This is in agreement with observations made on actual physical systems (Réocreux 1974). As it is well known from the theory of determinants, once, for some $i$, two distinct relations $\Delta = 0$ and $N_i = 0$, (with $N_i$ neither identical to zero nor divisible by $\Delta$), are satisfied, all the other $N_i$ vanish. Therefore, the compatibility condition is generally given by an equation $N_i = 0$.

*Remarks.* Removing the indeterminacy involves calculation of the limits of $N_i/\Delta$. In some cases—e.g. see section 3.1.1.—and for some $x_i$, this calculation is quite easy. It is important to notice that in several cases—see section 3.1.2.—some of the $N_i$ are identical to zero.

If the compatibility condition is not satisfied when $\Delta = 0$, then the problem is *impossible*. Impossibility means that some of the values assumed for the $x_i$ at the inlet are not realistic.

If all the $N_i$ are equal to zero when $\Delta = 0$, then the just-mentioned indeterminacy occurs, and two situations must be distinguished, depending on the form of the $N_i$. The determinant $N_i$ is *like* $\Delta$, a function of the L.H.S. coefficients ($x_i$, $\eta_i$, $\zeta_i$, ...) and, *unlike* $\Delta$, a function of R.H.S. quantities ($A'/A$, $g \cos\theta$, $\tau_0$, $q_0$, ...). Since all these coefficients and quantities are known functions of the $x_i$ and of $z$, the determinants $\Delta$ and $N_i$ are functions of the $x_i$ and of $z$.

In some particular cases, for any $i$, the quantities

$$f_i = \frac{N_i}{\Delta}$$

are always well-defined, even in the sections where $\Delta = 0$. Here, $N_i$ must be zero as soon as $\Delta = 0$, and the compatibility condition is identically verified (For example, all are zero because of a common factor). The gradients

$$\frac{dx_i}{dz} = \frac{N_i}{\Delta} = f_i$$

remain defined in any section where $\Delta = 0$. The inlet value of any of the $x_i$ may be slightly modified without inducing impossibility. In particular, the flow-rate can be increased. The only noticeable effect of such a change is to shift or to eliminate the indetermination section. The flow cannot be termed critical in the sense implied in the introduction of this paper.

Usually, if the compatibility condition is verified when $\Delta = 0$, it is not identically, i.e. $N_i = 0$ due to the local values of the previously-mentioned R.H.S. quantities. Generally, in such a case, depending on its sign, a slight modification of one of the $x_i$'s inlet value either induces impossibility (the $\Delta = 0$ and $N_i = 0$ cross-sections being differently shifted) or eliminates the indeterminacy (the $\Delta = 0$ cross-section being eliminated): the flow is said to be critical. The corresponding section is the critical section. According to the physical experience, the flow-rate can be decreased (eliminating the indeterminacy) but not increased (inducing impossibility) when the flow is critical, the inlet conditions except the flow-rate remaining unchanged.

*To summarize*:

$$\Delta = 0 \qquad [14]$$

is a necessary critical flow criterion. It involves only the L.H.S. of the equations and in particular neither $A'/A$ nor $\theta$. Its study *must* be complemented by that of the compatibility condition:

$$N_i = 0 \qquad [15]$$

where the choice of $i$ may be subject to some restrictions. The compatibility condition involves R.H.S. quantities (and among them $A'/A$ in general). It allows determination of the actual character of the flow, (i.e. critical or not) and, if the flow is critical, of the critical section. In single phase, horizontal nozzle flow without friction nor heat transfer, it explains, for instance, why the critical section is the throat of the nozzle.

2.3. *Propagation of small disturbances*

The application to transients of the procedure used to introduce the critical flow condition, i.e. trying to calculate the values of the dependent variables, leads directly to the theory of characteristics (Courant & Friedrichs 1961).

The dependent variables $x_i$ being given along a curve, tangent to the direction $V$ in the $(t, z)$ plane, the set of partial differential equations is complemented by the relations expressing the total derivatives of the $x_i$ in the direction $V$, i.e.

$$\frac{\mathrm{D}x_i}{\mathrm{D}t} = \frac{\partial x_i}{\partial t} + V\frac{\partial x_i}{\partial z}.$$

In the set of equations, the partial derivatives, with respect to one of the two independent variables, can then be expressed in terms of the known total derivatives and of the partial derivatives with respect to the other independent variable (for instance $z$). For single-phase flow a set of three linear algebraic equations with three unknowns,

$$\frac{\partial T}{\partial z}, \quad \frac{\partial w}{\partial z}, \quad \frac{\partial s}{\partial z},$$

is obtained. For two-phase flow, a set of six linear algebraic equations with six unknowns,

$$\frac{\partial \alpha_G}{\partial z}, \quad \frac{\partial p}{\partial z}, \quad \frac{\partial w_L}{\partial z}, \quad \frac{\partial \Delta w}{\partial z}, \quad \frac{\partial \Delta h_G}{\partial z}, \quad \frac{\partial \Delta h_L}{\partial z},$$

is obtained.

In each case, the characteristic equation is written by equating to zero the determinant of the set. Solved for $V$, it gives the displacement velocities of small disturbances. It can be easily verified that the constant term in the characteristic equation is the determinant $\Delta$ of the set used in section 2.2. Hence, when $\Delta = 0$, one of the roots of the characteristic equation is $V = 0$, i.e. a stationary small disturbance can exist. The detailed analysis of the other roots of the characteristic equation is presented elsewhere (see hereunder and Fritte 1974).

## 3. REVIEW OF THE SINGLE-PHASE FLOW CASE

### 3.1. *Critical condition*

The critical flow criterion [14] involves the determinant $\Delta$ of the set of single phase flow equations written for steady state. $\Delta$ is a third order determinant. Equation [14] can be written:

$$\Delta \equiv \begin{vmatrix} w\rho'_{T,s} & \rho & w\rho'_{s,T} \\ \rho'_{T,s}(a_s^2-\eta_T) & \rho w(1-\eta_w) & \rho'_{s,T}(a_T^2-\eta_s) \\ w\rho'_{T,s}(\zeta_T+\eta_T) & w^2\rho(\zeta_w+\eta_w) & w\rho'_{s,T}[(\rho T/\rho'_{s,T})+\zeta_s+\eta_s] \end{vmatrix} = 0,$$

or by developing and dividing by $w$, which eliminates the root $w=0$:

$$\frac{\Delta}{\rho\rho'_{T,s}\rho'_{s,T}w} \equiv [(w^2-a_s^2)+(\eta_T-w^2\eta_w)]\frac{\rho T}{\rho'_{s,T}}+[(w^2-a_s^2)+(\eta_T-w^2\eta_w)](\zeta_s+\eta_s)$$
$$-[(w^2-a_T^2)+(\eta_s-w^2\eta_w)](\zeta_T+\eta_T)+w^2[(a_s^2-a_T^2)+(\eta_s-\eta_T)](\zeta_w+\eta_w)=0. \quad [16]$$

This is the general expression of the critical flow criterion. It can be used to study any particular case. To demonstrate the possibilities of the method, three examples are studied hereunder. In the first of them, the $\eta_i$ and $\zeta_i$, i.e. the external constitutive laws, are given. In the other two, the evolution of the fluid is given.

#### 3.1.1. *Cases with differential terms in the external constitutive laws*

For illustration purpose, it is sufficient to take into account a rather limited number of parameters $\eta_i$ and $\zeta_i$. Assuming $\eta_T=\eta_w=\eta_s=0$, [16] can be written:

$$w^2=a_s^2-(a_s^2-a_T^2)\frac{w^2\zeta_w-\zeta_T}{\dfrac{\rho T}{\rho'_{s,T}}+\zeta_s-\zeta_T}. \quad [17]$$

3.1.1.1. When

$$\zeta_T=w^2\zeta_w\neq\zeta_s+\frac{\rho T}{\rho'_{s,T}}, \quad [18]$$

which is in particular the case when $\zeta_T=\zeta_w=\zeta_s=0$ [17] is satisfied for

$$w=w_c=\pm a_s$$

(isentropic sound velocity). The expressions of the determinants $N_i$ can be written:

$$\frac{N_T}{\rho\rho'_{s,T}}=w\left(\frac{\rho T}{\rho'_{s,T}}+\zeta_s\right)\left(\tau_0+\rho g\cos\theta-\rho w^2\frac{A'}{A}\right)-w\zeta_T\left(\tau_0+\rho g\cos\theta-\rho a_T^2\frac{A'}{A}\right)$$
$$-(q_0+w\tau_0)(w^2-a_T^2),$$

$$\frac{N_w}{w\rho'_{T,s}\rho'_{s,T}}=w\left(\frac{\rho T}{\rho'_{s,T}}+\zeta_s\right)\left(\rho a_s^2\frac{A'}{A}-\tau_0-\rho g\cos\theta\right)-w\zeta_T\left(\rho a_T^2\frac{A'}{A}-\tau_0-\rho g\cos\theta\right)$$
$$+(q_0+w\tau_0)(a_s^2-a_T^2),$$

$$\frac{N_s}{\rho\rho'_{T,s}}=\left(\rho w\frac{A'}{A}\zeta_T+q_0+w\tau_0\right)(w^2-a_s^2).$$

The indeterminacy of $\mathrm{d}s/\mathrm{d}z$ can be easily removed, since everywhere along the pipe, and in particular in the critical section:

$$\frac{\mathrm{d}s}{\mathrm{d}z}=\frac{\rho w\dfrac{A'}{A}\zeta_T+q_0+w\tau_0}{w\rho'_{s,T}\left(\dfrac{\rho T}{\rho'_{s,T}}+\zeta_s-\zeta_T\right)}.$$

This shows that the evolution of the fluid is *not* isentropic in general. It is isentropic if and only if:

$$\rho w\frac{A'}{A}\zeta_T+q_0+w\tau_0=0.$$

Such is the case within the framework of the current assumption $\zeta_T=0$ when $q_0+w\tau_0=0$. This implies through [18] that $\zeta_w=0$ and $\zeta_s\neq\rho T/\rho'_{s,T}$, conditions which are compatible but a little less restrictive than the current assumptions $\zeta_T=\zeta_w=\zeta_s=0$.

Since $N_s=0$ when $\Delta=0$, it cannot be taken as the compatibility relationship. This compatibility relationship is the vanishing condition of $N_T$ and $N_w$, and it enables the critical section to be determined:

$$\frac{A'}{A}=\frac{w_c(\tau_0+\rho g\cos\theta)\left(\dfrac{\rho T}{\rho'_{s,T}}+\zeta_s-\zeta_T\right)-(q_0+w\tau_0)(a_s^2-a_T^2)}{\rho w_c\left[a_s^2\left(\dfrac{\rho T}{\rho'_{s,T}}+\zeta_s\right)-a_T^2\zeta_T\right]}. \qquad [19]$$

3.1.1.2. When

$$\frac{\rho T}{\rho'_{s,T}}+\zeta_s=w^2\zeta_w\neq\zeta_T, \qquad [20]$$

[17] is satisfied for

$$w=w_c=\pm a_T.$$

The expressions of the determinants $N_i$ can be written:

$$\frac{N_T}{\rho\rho'_{s,T}}=-(w^2-a_T^2)\left(q_0+w\tau_0+\rho w^3\frac{A'}{A}\zeta_w\right),$$

$$\frac{N_w}{w\rho'_{T,s}\rho'_{s,T}}=\rho w\frac{A'}{A}[a_s^2w^2\zeta_w-a_T^2\zeta_T]+w(\tau_0+\rho g\cos\theta)(\zeta_T-w^2\zeta_w)+(q_0+w\tau_0)(a_s^2-a_T^2),$$

$$\frac{N_s}{\rho\rho'_{T,s}}=\rho w^3\frac{A'}{A}[\zeta_T-a_s^2\zeta_w]+w(\tau_0+\rho g\cos\theta)(w^2\zeta_w-\zeta_T)+(q_0+w\tau_0)(w^2-a_s^2).$$

The indeterminacy of $\mathrm{d}T/\mathrm{d}z$ can be easily removed, since everywhere along the pipe, and in particular in the critical section:

$$\frac{\mathrm{d}T}{\mathrm{d}z}=-\frac{\rho w^3\dfrac{A'}{A}\zeta_w+q_0+w\tau_0}{w\rho'_{T,s}(w^2\zeta_w-\zeta_T)}.$$

The evolution of the fluid is *not* isothermal in general. It is isothermal if and only if:

$$\rho w^3 \frac{A'}{A} \zeta_w + q_0 + w\tau_0 = 0.$$

Such is the case within the framework of the current assumption $\zeta_w = 0$ when $q_0 + w\tau_0 = 0$. Further, as $\zeta_T \neq 0$, these assumptions are *not* compatible with the current assumptions $\zeta_T = \zeta_w = \zeta_s = 0$.

Since $N_T = 0$ when $\Delta = 0$, it cannot be taken as the compatibility relationship. This compatibility relationship is the vanishing condition of $N_w$ and $N_s$, and it enables the critical section to be determined: it has the same analytical expression as [19].

3.1.1.3. When

$$\zeta_T < w^2\zeta_w < \frac{\rho}{\rho'_{s,T}} + \zeta_s, \quad \text{or} \quad \frac{\rho T}{\rho'_{s,T}} + \zeta_s < w^2\zeta_w < \zeta_T,$$

[17] is satisfied for $w = w_c$, where

$$a_T < |w_c| < a_s.$$

By the extension of the ideal gas results, this value could be called a "polytropic sound velocity".

3.1.2. *Isentropic flow*

This case is dealt with here to stress the fact that, when the evolution of the fluid is postulated as a substitute for the unknown transfer laws, fundamental interpretation problems are encountered. These problems can only be solved if the transfer terms involve derivatives.

The flow is isentropic if and only if the condition $\mathrm{d}s/\mathrm{d}z = 0$ is contained in the solution of the set of equations. This implies that the condition $N_s = 0$, involving the R.H.S. of the equations, is satisfied everywhere along the pipe, i.e.:

$$N_s\left(x_i, \eta_i, \zeta_i, \frac{A'}{A}, g \cos\theta, \tau_0, q_0\right) \equiv 0. \qquad [21]$$

The physical significance of this condition is that, for an isentropic flow to be achieved (with a given fluid, a given geometry and a given range of $x_i$) a constraint [21] has to be satisfied by the external constitutive laws, i.e. by both the boundary conditions and the transfer laws. It is satisfied for instance in the classical case $\eta_i = \zeta_i = 0$ when $\tau_0 = q_0 = 0$.

In all the sections where $\Delta \neq 0$, and because $N_s \equiv 0$, the set may be considered as a compatible set of 3 equations with only 2 unknowns, and it is equivalent to any subset of 2 of the equations, provided the determinant of this subset is not zero (Bouré & Réocreux 1972). The same is true by continuity in any isolated section where $\Delta$ vanishes. Since the solution may be determined with any of the above subsets, the discussion of section 2.2. must be applied to these subsets. The consequences are:

(a) For the flow to be critical, it is necessary (see Réocreux 1974, p. I, 111), that the determinants $\delta_s^i$ of *all* the above subsets vanish simultaneously, $\delta_s^i$ being obtained from $\Delta$ on removing the column of the coefficients of $\mathrm{d}s/\mathrm{d}z$ and the $i$th row.

Conversely, the theory of determinants shows that if, in a given cross-section, two of the subset determinants $\delta_s^i$ and $\delta_s^j$ vanish, $i$ and $j$ being arbitrary, except for some particular cases, the other subset determinant also vanishes, $\Delta = 0$, and $N_s = 0$.

Accordingly, *for an isentropic flow*, [14] may be replaced by more specific necessary critical flow criteria:

$$\delta_s^i = 0, \quad \delta_s^j = 0, \qquad [22]$$

where the choice of $i$ and $j$ $(i \neq j)$ may be subject to some restrictions.

(b) When criteria [22] are verified, the compatibility conditions of all the subsets are equivalent in general. One of these compatibility conditions (an arbitrary one, except for some particular cases) must be used instead of [15].

The necessary critical flow criteria [22] may be written:

$$\begin{vmatrix} w\rho'_{T,s} & \rho \\ \rho'_{T,s}(a_s^2-\eta_T) & \rho w(1-\eta_w) \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} w\rho'_{T,s} & \rho \\ w\rho'_{T,s}(\zeta_T+\eta_T) & w^2\rho(\zeta_w+\eta_w) \end{vmatrix} = 0.$$

They yield:

$$w^2 - a_s^2 = w^2\eta_w - \eta_T, \tag{23}$$

$$w^2(\zeta_w + \eta_w) = \zeta_T + \eta_T. \tag{24}$$

The compatibility condition may be taken as

$$\begin{vmatrix} -\rho w(A'/A) & \rho \\ -\tau_0 - \rho g \cos\theta & \rho w(1-\eta_w) \end{vmatrix} = 0,$$

i.e.

$$\rho w^2(1-\eta_w)\frac{A'}{A} = \tau_0 + \rho g \cos\theta. \tag{25}$$

Equations [23] and [24] are the critical flow conditions. Equation [25] determines the position of the critical section. The results of section 3.1.1.1. for an isentropic flow appear as a particular case of the foregoing.

Equations [23] and [24] show that, *for an isentropic flow to be critical, a necessary condition* on the $\eta_i$ and $\zeta_i$ is

$$\frac{\zeta_T + \eta_T}{\zeta_w + \eta_w} = \frac{a_s^2 - \eta_T}{1 - \eta_w}.$$

When this is satisfied

$$w_c^2 = \frac{a_s^2 - \eta_T}{1 - \eta_w}.$$

Thus, when the $\eta_i$ and $\zeta_i$ are not known, [22] may be interpreted as consisting of a relationship between the $\eta_i$ and $\zeta_i$, and an actual critical flow criterion (here, [24] happens to correspond to [18], and [23] to [17]). If the isentropic flow critical phenomenon is considered as well known, the above equations give information on the external constitutive laws. For instance, $w_c = \pm a_s$ implies through [23] and [24] that in the critical section:

$$\eta_T = w^2\eta_w \quad \text{and} \quad \zeta_T = w^2\zeta_w.$$

3.1.3. *Isothermal flow*

The discussion parallels that of section 3.1.2. The equations giving the critical velocity and the condition verified by the external constitutive laws in the critical section are:

$$w^2 - a_T^2 = w^2\eta_w - \eta_s, \tag{26}$$

$$w^2(\zeta_w + \eta_w) = \frac{\rho T}{\rho'_{s,T}} + \zeta_s + \eta_s. \qquad [27]$$

The critical section is determined by [25]. The results of section 3.1.1.2. for an isothermal flow appear as a particular case of the foregoing.

3.1.4. *Discussion*

The possibility of generalizing, i.e. of obtaining critical velocities different from $a_s$ and dealing with any flow evolution, is a consequence of the presence of differential terms in the external constitutive laws. Taking these terms into account is not common practice. It is pointed out, however, that some authors (see Brun *et al.* 1968) implicitly use differential terms, in $\mathcal{Q}$, to study isothermal flows.

Without differential terms, the classical result is always obtained: the velocity of the fluid in the critical section is equal to the isentropic velocity of sound, whatever the evolution of the fluid may be. To explain deviations from this isentropic sound velocity, a possibility is therefore the presence of differential terms in the constitutive laws.

3.2. *Propagation of small disturbances*

The characteristic equation (see section 2.3.) is:

$$\begin{vmatrix} \rho'_{T,s}(w-V) & \rho & \rho'_{s,T}(w-V) \\ \rho'_{T,s}\left[a_s^2 - \frac{\eta_T}{w}(w-V)\right] & \rho(1-\eta_w)(w-V) & \rho'_{s,T}\left[a_T^2 - \frac{\eta_s}{w}(w-V)\right] \\ \rho'_{T,s}(\zeta_T+\eta_T)(w-V) & \rho w(\zeta_w+\eta_w)(w-V) & \rho'_{s,T}\left[\frac{\rho T}{\rho'_{s,T}} + \zeta_s + \eta_s\right](w-V) \end{vmatrix} = 0.$$

$V = w$ (material velocity) is always a root of this equation. The two others are given by

$$\left[(w-V)^2 - a_s^2 + \frac{\eta_T}{w}(w-V) - \eta_w(w-V)^2\right]\left(\frac{\rho T}{\rho'_{s,T}} + \zeta_s + \eta_s\right)$$
$$- \left[(w-V)^2 - a_T^2 + \frac{\eta_s}{w}(w-V) - \eta_w(w-V)^2\right](\zeta_T + \eta_T)$$
$$+ w(w-V)\left[a_s^2 - a_T^2 + \frac{\eta_s - \eta_T}{w}(w-V)\right](\zeta_w + \eta_w) = 0, \qquad [28]$$

which may be compared to [16]. Developing [28] yields:

$$V^2\left[(1-\eta_w)\left(\frac{\rho T}{\rho'_{s,T}} + \zeta_s - \zeta_T\right) + (\eta_s - \eta_T)(1+\zeta_w)\right]$$
$$- V\left[2w(1-\eta_w)\left(\frac{\rho T}{\rho'_{s,T}} + \zeta_s - \zeta_T\right) + 2w(\eta_s - \eta_T)(1+\zeta_w)\right.$$
$$\left. + \frac{\eta_T}{w}\left(\frac{\rho T}{\rho'_{s,T}} + \zeta_s\right) - \frac{\eta_s}{w}\zeta_T + w(a_s^2 - a_T^2)(\zeta_w + \eta_w)\right] + \frac{\Delta}{\rho\rho'_{T,s}\rho'_{s,T}w} = 0. \qquad [29]$$

When the flow is critical, $\Delta = 0$ in the critical section and, as already pointed out in section 2.3., $V = 0$ is a root of the characteristic equation. The remaining root can be computed through [29]. It is

$$V_c = 2w_c + \frac{\frac{\eta_T}{w_c}\left(\frac{\rho T}{\rho'_{s,T}} + \zeta_s\right) - \frac{\eta_s}{w_c}\zeta_T + w_c(a_s^2 - a_T^2)(\zeta_w + \eta_w)}{(1-\eta_w)\left(\frac{\rho T}{\rho'_{s,T}} + \zeta_s - \zeta_T\right) + (\eta_s - \eta_T)(1+\zeta_w)}. \qquad [30]$$

It results from the interpretation of the single-phase flow critical conditions, as given in the introduction, that $V_c$ and $w_c$ must have the same sign, so that disturbances initiated downstream of the critical section cannot propagate upstream of this section.

Only some of the examples developed in section 3.1.1. are studied hereunder.

3.2.1. *No differential term in the external constitutive laws*

When all the $\eta_i$ and $\zeta_i$ are zero, the roots of the characteristic equation in the critical section are

$$0, \quad w_c, \quad 2w_c.$$

This is the classical result.

3.2.2. *Some particular cases with differential terms in the external constitutive laws*

Assuming $\eta_T = \eta_w = \eta_s = 0$, [30] can be written:

$$V_c = 2w_c + w_c \frac{(a_s^2 - a_T^2)\zeta_w}{\dfrac{\rho T}{\rho'_{s,T}} + \zeta_s - \zeta_T}. \tag{31}$$

3.2.2.1. When

$$\zeta_T = w^2\zeta_w \neq \zeta_s + \frac{\rho T}{\rho'_{s,T}}, \tag{18}$$

$w_c = \pm a_s$ and the roots of the characteristic equation in the critical section are

$$0, \quad w_c, \quad w_c\left[2 + \frac{\gamma - 1}{\gamma}\frac{\zeta_T}{\dfrac{\rho T}{\rho'_{s,T}} + \zeta_s - \zeta_T}\right] \quad \text{with} \quad \gamma = \frac{a_s^2}{a_T^2}.$$

Since they must have the same sign, $\zeta_T$ must *not* be comprised between

$$\frac{\rho T}{\rho'_{s,T}} + \zeta_s \quad \text{and} \quad \frac{2\gamma}{\gamma + 1}\left(\frac{\rho T}{\rho'_{s,T}} + \zeta_s\right).$$

Such is of course the case within the framework of the current assumption $\zeta_s = 0$, $\zeta_T = 0$.

3.2.2.2. When

$$\frac{\rho T}{\rho'_{s,T}} + \zeta_s = w^2\zeta_w \neq \zeta_T, \tag{20}$$

$w_c = \pm a_T$ and the roots of the characteristic equation in the critical section are

$$0, \quad w_c, \quad w_c\left[2 + (\gamma - 1)\frac{a_T^2\zeta_w}{a_T^2\zeta_w - \zeta_T}\right].$$

Since they must have the same sign, $\zeta_T$ must *not* be comprised between $a_T^2\zeta_w$ and $[(\gamma + 1)/2]a_T^2\zeta_w$.

3.2.3. *Discussion*

The presence of differential terms in the expressions of the external constitutive laws is fully compatible with the classical interpretation of the single-phase critical phenomenon in terms of propagation of small disturbances. The study of the propagation velocities discloses however that the values of the $\eta_i$ and $\zeta_i$ are subject to one restriction: the R.H.S. of [30] must have the sign of $w_c$.

## 4. TWO-PHASE FLOWS

### 4.1. *Critical condition*

The critical flow criterion [14] involves the determinant $\Delta$ of the set of equations written for steady state. $\Delta$ is a 6th order determinant. The resulting equation, [32], is given in figure 2.

This general expression of the critical flow criterion enables any particular case to be studied. Two models have been chosen to demonstrate the possibilities of the method: a two-velocity-two-temperature model (Réocreux 1974), where the critical flow criterion is given by the vanishing condition of a sixth order determinant, and a partial non-equilibrium model (Giot & Fritte 1971), involving a fifth order determinant. This choice is given only for purposes of illustration and is more or less arbitrary. It does not lay any claim as to the actual value of the models. Other models are considered in Bouré *et al.* (1975).

As for single-phase flows, two different points of view are adopted: firstly, particular sets of values of the $\lambda_i^z$, $\mu_i^z$ and $\nu_i^z$ are selected. Secondly, the evolution of the fluid is prescribed.

#### 4.1.1. *Selected values of the coefficients* $\lambda_i^z$, $\mu_i^z$, $\nu_i^z$

4.1.1.1. *Two-velocity-two-temperature model, with no differential terms in the constitutive laws.* All the $\lambda_i^z$, $\mu_i^z$ and $\nu_i^z$ are zero. Equations [11–13] are reduced to:

$$M = M_0, \quad (MV) = (MV)_0, \quad (MH) = (MH)_0$$

which are functions of $z$, $\alpha_G$, $p$, $w_L$, $\Delta w$, $\Delta h_G$ and $\Delta h_L$ only.

Introducting these conditions in the criterion [32] and developing the determinant $\Delta$ yields:

$$\alpha_G^2\alpha_L^2\rho_G^2\rho_L^2 w_G w_L \left\{\alpha_L\rho_G w_G^2 + \alpha_G\rho_L w_L^2 - w_G^2 w_L^2\left[\alpha_L\rho_G\left(\frac{\partial\rho_L}{\partial p}\right)_{s_L} + \alpha_G\rho_L\left(\frac{\partial\rho_G}{\partial p}\right)_{s_G}\right]\right\} = 0, \tag{33}$$

where the following thermodynamic relationship has been used:

$$\rho_K\left(\frac{\partial\rho_K}{\partial p}\right)_{s_K} = \left(\frac{\partial\rho_K}{\partial h_K}\right)_p + \rho_K\left(\frac{\partial\rho_K}{\partial p}\right)_{h_K}.$$

The critical flow criterion [33] can be expressed in terms of the square of the mass flux density:

$$G^2 = \frac{\dfrac{\alpha_G^3\rho_G}{x^2} + \dfrac{\alpha_L^3\rho_L}{(1-x)^2}}{\dfrac{\alpha_G}{\rho_G}\left(\dfrac{\partial\rho_G}{\partial p}\right)_{s_G} + \dfrac{\alpha_L}{\rho_L}\left(\dfrac{\partial\rho_L}{\partial p}\right)_{s_L}}, \tag{34}$$

the quality $x$ being defined as the ratio $\alpha_G\rho_G w_G/G$. The rather complicated compatibility condition enables the critical section to be determined.

4.1.1.2. *Partial non-equilibrium model.* For the $\nu_i^z$, the following equalities and inequality are assumed:

$$\nu_\alpha^z = -\rho_G w_G\left(h_G + \frac{w_G^2}{2}\right),$$

$$\nu_p^z = -\alpha_G w_G\left[\left(h_G + \frac{w_G^2}{2}\right)\rho'_{G/p,\Delta} + \rho_G h'_{G\text{sat}}\right],$$

$$\Delta \equiv \begin{vmatrix}
\rho_G w_G + \lambda_\alpha^z & \alpha_G w_G \rho'_{G/p,\Delta} + \lambda_p^z & \alpha_G \rho_G + \lambda^z_{wL} & \alpha_G \rho_G + \lambda^z_{\Delta w} & \alpha_G w_G \rho'_{G/h,p} + \lambda^z_{hG} & \lambda^z_{hL} \\
-\rho_L w_L - \lambda_\alpha^z & \alpha_L w_L \rho'_{L/p,\Delta} - \lambda_p^z & \alpha_L \rho_L - \lambda^z_{wL} & -\lambda^z_{\Delta w} & -\lambda^z_{hG} & \alpha_L w_L \rho'_{L/h,p} - \lambda^z_{hL} \\
\rho_G w_G^2 + \mu_\alpha^z & \alpha_G(1 + w_G^2 \rho'_{G/p,\Delta}) + \mu_p^z & 2\alpha_G \rho_G w_G + \mu^z_{wL} & 2\alpha_G \rho_G w_G + \mu^z_{\Delta w} & \alpha_G w_G^2 \rho'_{G/h,p} + \mu^z_{hG} & \mu^z_{hL} \\
-\rho_L w_L^2 - \mu_\alpha^z & \alpha_L(1 + w_L^2 \rho'_{L/p,\Delta}) - \mu_p^z & 2\alpha_L \rho_L w_L - \mu^z_{wL} & -\mu^z_{\Delta w} & -\mu^z_{hG} & \alpha_L w_L^2 \rho'_{L/h,p} - \mu^z_{hL} \\
\rho_G w_G \left(h_G + \frac{w_G^2}{2}\right) + \nu_\alpha^z & \alpha_G w_G \left[\left(h_G + \frac{w_G^2}{2}\right) \rho'_{G/p,\Delta} + \rho_G h'_{Gsat}\right] + \nu_p^z & \alpha_G \rho_G \left(h_G + \frac{3w_G^2}{2}\right) + \nu^z_{wL} & \alpha_G \rho_G \left(h_G + \frac{3w_G^2}{2}\right) + \nu^z_{\Delta w} & \alpha_G w_G \left[\left(h_G + \frac{w_G^2}{2}\right) \rho'_{G/h,p} + \rho_G\right] + \nu^z_{hG} & \nu^z_{hL} \\
-\rho_L w_L \left(h_L + \frac{w_L^2}{2}\right) - \nu_\alpha^z & \alpha_L w_L \left[\left(h_L + \frac{w_L^2}{2}\right) \rho'_{L/p,\Delta} + \rho_L h'_{Lsat}\right] - \nu_p^z & \alpha_L \rho_L \left(h_L + \frac{3w_L^2}{2}\right) - \nu^z_{wL} & -\nu^z_{\Delta w} & -\nu^z_{hG} & \alpha_L w_L \left[\left(h_L + \frac{w_L^2}{2}\right) \rho'_{L/h,p} + \rho_L\right] - \nu^z_{hL}
\end{vmatrix} = 0 \qquad [32]$$

Figure 2. Two-phase flow: critical flow criterion.

$$\nu^z_{wL} = \nu^z_{\Delta w} = -\alpha_G\rho_G\left(h_G + \frac{3w_G^2}{2}\right),$$

$$\nu^z_{hL} = 0,$$

$$\nu^z_{hG} \neq -\alpha_G w_G\left[\left(h_G + \frac{w_G^2}{2}\right)\rho'_{G/h,p} + p_G\right].$$

The energy equation of the gas phase written for steady state becomes:

$$\left\{\alpha_G w_G\left[\left(h_G + \frac{w_G^2}{2}\right)\rho'_{G/h,p} + p_G\right] + \nu^z_{hG}\right\}\frac{d(\Delta h_G)}{dz} = (MH)_0 + q_{G,0} - \alpha_G\rho_G w_G g \cos\theta - \alpha_G\rho_G w_G\left(h_G + \frac{w_G^2}{2}\right)\frac{A'}{A}. \quad [35]$$

On developing the determinant $\Delta$, one obtains:

$$\Delta = \left\{\alpha_G w_G\left[\left(h_G + \frac{w_G^2}{2}\right)\rho'_{G/h,p} + p_G\right] + \nu^z_{hG}\right\}\Delta_5$$

where $\Delta_5$ is the determinant of a fifth order subset. The critical flow criterion $\Delta = 0$ is thus equivalent to $\Delta_5 = 0$. The compatibility condition enables the critical section to be determined.

In particular if the R.H.S. member of [35] is assumed to vanish then $(h_G - h_{Gsat})$ remains constant. If moreover $\lambda_\alpha^z = \lambda_p^z = \lambda^z_{wL} = \lambda^z_{\Delta w} = \lambda^z_{hL} = \mu_\alpha^z = \mu_p^z = \mu^z_{wL} = \mu^z_{\Delta w} = \mu^z_{hL} = 0$, and $h_G = h_{Gsat}$, then the critical flow condition $\Delta_5 = 0$ takes the form of one of those proposed in the literature (Giot & Fritte 1971). The determinant $\Delta_5$ no longer involves any of the $\lambda_i$, $\mu_i$, $\nu_i$. The elements of its last row are identical to those obtained by summing up the energy equations of the gas and liquid phases: the result is the same as if a single mixture energy equation were used instead of two phasic energy equations. It can be noticed that the same result is obtained with at least another set of values for the $\nu_i$, i.e.

$$\left[\nu_\alpha^z = -\rho_L w_L\left(h_L + \frac{w_L^2}{2}\right),\quad \nu_p^z = \alpha_L w_L\left[\left(h_L + \frac{w_L^2}{2}\right)\rho'_{L/p,\Delta} + \rho_L h'_{Lsat}\right], \ldots\right].$$

4.1.2. *Prescribed evolution*

As in the case of single phase flow (section 3.1.2) the consequences of prescribing an evolution of the fluid must be investigated.

This study is restricted to the example of flows with constant non-equilibrium of the gas phase:

$$\frac{d(\Delta h_G)}{dz} = 0.$$

If such a condition is contained in the solution of the set of equations, it implies that a condition $N_{\Delta hG} = 0$, involving the R.H.S. of the equations is satisfied everywhere along the pipe, i.e.:

$$N_{\Delta hG}\left(x_i, \lambda_i^z, \mu_i^z, \nu_i^z, M_0, (MV)_0, (MH)_0, \frac{A'}{A}, g\cos\theta, \tau_{K,0}, q_{K,0}\right) \equiv 0. \quad [36]$$

The physical significance of this condition is that, for constant non-equilibrium to be achieved (with a given fluid, a given geometry and a given range of $x_i$), a constraint [36] has to be satisfied by the external constitutive laws and the interfacial transfer laws.

In all the sections where $\Delta \neq 0$, and because $N_{\Delta hG} \equiv 0$, the set may be considered as a compatible set of six equations with only five unknowns, and it is equivalent to any subset of five of the equations provided the determinant of this subset is not zero. The same is true by continuity in any isolated section where $\Delta$ vanishes. Since the solution may be determined with any of the above subsets, the discussion of section 2.2. (critical flow condition) must be applied to these subsets. The consequences are:

(a) For the flow to be critical, it is necessary that the determinants $\Delta^i_{\Delta hG}$ of *all* the above subsets vanish simultaneously, $\Delta^i_{\Delta hG}$ being obtained from $\Delta$ on removing the column of the coefficients of $d\Delta h_G/dz$ and the $i$th row.

Conversely, the theory of determinants shows that if, in a given cross-section, two of the subset determinants $\Delta^i_{\Delta hG}$ and $\Delta^j_{\Delta hG}$ vanish, $i$ and $j$ being suitably chosen, the other subset determinants also vanish, $\Delta = 0$ and $N_{\Delta hG} = 0$.

Accordingly, for a flow with constant non-equilibrium of the gas phase, the critical flow criterion [14] may be replaced by more specific necessary critical flow criteria

$$\Delta^i_{\Delta hG} = 0, \quad \Delta^j_{\Delta hG} = 0. \qquad [37]$$

(b) When [37] are verified, the compatibility conditions of all the subsets are equivalent in general. One of these compatibility conditions must be used instead of [15].

The results of section 4.1.1.2 for a flow with constant non-equilibrium of the gas phase appear as a particular case of the foregoing. Like criteria [22] for single-phase flows, criteria [37] may be interpreted as consisting of a relationship between the $\lambda_i^z$, $\mu_i^z$, $\nu_i^z$, (corresponding to the assumptions of section 4.1.1.2) and an actual critical flow criterion (corresponding to $\Delta_5 = 0$ in section 4.1.1.2).

4.1.3. *Discussion*

The conclusion is analogous to that of section 3.1.4. for single-phase flows: the same result (section 4.1.1.1) is always obtained for two-phase flows when the presence of differential terms in the constitutive laws is not accepted. However, when for single-phase flows the isentropic sound velocity is in general a very good approximation to the actual critical velocity, the result of section 4.1.1.1 for two-phase flows is bad. This is true both practically, since it predicts critical velocities which are several times the actual ones, and theoretically, since it does not allow for any influence of the interfacial transfers on the critical phenomenon (Réocreux 1974).

To obtain other results, and in particular to find and to justify existing models whose results are closer to the experimental data, a possibility is to allow for the presence of differential terms in the constitutive laws. Examples have been given in sections 4.1.1. and 4.1.2., with differential terms present only in the interfacial transfer laws. Any model, if consistent, appears as a particular case of the model used here, complemented by appropriate assumptions on the constitutive laws. Since the six phasic conservation equations, based on the conservation laws, have to be satisfied anyway, this result is entirely satisfactory.

Although occasionally differential terms have already been used (see for instance Wallis 1969 and Ivandaev & Nigmatulin 1972) their importance does not seem to have been fully realized. Physically, their influence proceeds from the fact that they strongly affect the coupling between the phases.

4.2. *Propagation of small disturbances*

This section should parallel section 3.2. However, it is not possible to give here the most general characteristic equation (corresponding to [28] or [29] of the single-phase case) due to its size. The L.H.S. of this equation is a polynomial in $V$ of the sixth order. When the flow is critical, $V = 0$ is a root of the characteristic equation in the critical section. Since there are 5 remaining roots (real or complex), it is not possible to obtain analytical expressions, corresponding to [30], for them. This difference is, however, due only to the increase of the order of the equation and, for two phase cocurrent flows, there is no reason, up to now, to dismiss the classical interpretation of critical flow conditions, as given in the introduction: a flow is choked when disturbances initiated downstream of some critical section cannot propagate upstream of this section.

Accordingly, when the roots $V_c$ of the characteristic equation are real, they must either be zero or have the same sign as $w_G$ and $w_L$. The presence, often found out in existing models, of one pair of conjugate complex roots raises a problem which is specific to two-phase flows and is still controversial (cf. for instance, round table RT-1 on momentum and heat transfer mechanisms in two-phase flow, held during the 5th international heat transfer conference—Tokyo, September 1974). They probably correspond also to the propagation of "something" (Fritte 1974) and their real part must also have the same sign as $w_G$ and $w_L$. As found for single-phase flows, these restrictions on the signs of the roots of the characteristic equation correspond to restrictions on the values of the $\lambda_i$, $\mu_i$, $\nu_i$.

Two examples are given hereunder.

4.2.1. *No differential terms in the constitutive laws*

The case when all the $\lambda_i$, $\mu_i$, $\nu_i$ are zero has been studied at length by Bouré (1973). Four roots of the characteristic equation are real and have, in the critical section, the same sign as $w_G$ and $w_L$; they are: $V_1 = 0$, $V_2 = w_G$, $V_3 = w_L$, $w_G + a_{Gs} < V_4 < w_L + a_{Ls}$, where $a_{Gs}$ and $a_{Ls}$ are the single-phase isentropic sound velocities in the gas and liquid phase, respectively. When the two remaining roots are complex and conjugate, which is always the case with current values of the critical parameters, their real part has the same sign as $w_G$ and $w_L$ (Fritte 1974).

4.2.2. *An example with differential terms in the constitutive laws*

A relatively simple example is obtained when only momentum is transferred at the interface, i.e. $M_0 = (MH)_0 = 0$, $\lambda_i = \nu_i = 0$. Moreover, all the $\mu_i$ are assumed to be zero, except $\mu_p{}^z$ and $\mu_p{}'$. Simple algebraic transformations on the determinant of the L.H.S. of the characteristic equation show that it has the roots $V = w_G$, $V = w_L$, and the roots of

$$\rho_L(w_L - V)^2\left\{\alpha_G[\rho'_{G/p,s}(w_G - V)^2 - 1] - (\mu_p{}^z - \mu_p{}'V)\left[1 - \frac{\rho'_{G/h,p}}{\rho_G}w_G(w_G - V)\right]\right\}$$
$$+ \rho_G(w_G - V)^2\left\{\alpha_L[\rho'_{L/p,s}(w_L - V)^2 - 1] + (\mu_p{}^z - \mu_p{}'V)\left[1 - \frac{\rho'_{L/h,p}}{\rho_L}w_L(w_L - V)\right]\right\} = 0, \quad [38]$$

which can be compared to [3] in Bouré (1973). Further simplification can be obtained by assuming that the terms containing $\rho'_{K/h,p}$ are negligible in this equation, which is true for water within a large range of pressures and velocities.

Divided by $\rho_G\rho_L$, [38] yields:

$$\left(\alpha_G\frac{\rho'_{G/p,s}}{\rho_G} + \alpha_L\frac{\rho'_{L/p,s}}{\rho_L}\right)(w_G - V)^2(w_L - V)^2 - \alpha_G\frac{(w_L - V)^2}{\rho_G} - \alpha_L\frac{(w_G - V)^2}{\rho_L}$$
$$- (\mu_p{}^z - \mu_p{}'V)\left[\frac{(w_L - V)^2}{\rho_G} - \frac{(w_G - V)^2}{\rho_L}\right] = 0,$$

or assuming low pressure ($\rho_G/\rho_L \ll 1$), significant void fractions and reasonable slip ratios,

$$\frac{(w_L - V)^2}{\rho_G}[\alpha_G\rho'_{G/p,s}(w_G - V)^2 - (\alpha_G + \mu_p{}^z) + \mu_p{}'V] = 0. \quad [39]$$

It has, twice, the root $V = w_L$, which thus appears to be triple in the complete characteristic equation.

When the flow is critical, $V = 0$ is a root of the characteristic equation in the critical section. Hence

$$\alpha_G \rho'_{G/p,s} w^2_{Gc} - (\alpha_G + \mu_p^{z}) = 0.$$

Let $r$ be defined by

$$r = \frac{\alpha_G + \mu_p^{z}}{\alpha_G} = \rho'_{G/p,s} w^2_{Gc} = \left(\frac{w_{Gc}}{a_{Gs}}\right)^2, \quad [40]$$

so that

$$\mu_p^{z} = -(1 - r)\alpha_G. \quad [41]$$

Experimental data on critical flow rates and slip ratios imply that $r$, comprised between 0 and 1, is significantly smaller than 1. Conversely, [40] shows that it is possible to fit the model to the experimental critical data by adjusting only the coefficient $r$.

The remaining root is easy to calculate from [39], taking [40] into account. It is

$$V = 2w_{Gc} - \frac{\mu_p^{t}}{\alpha_G \rho'_{G/p,s}}, \quad [42]$$

and has the sign of $w_{Gc}$ if, and only if

$$\frac{\mu_p^{t}}{w_{Gc}} < \frac{2\alpha_G}{a^2_{Gs}}. \quad [43]$$

To summarize, the roots of the characteristic equation in the critical section are $V_1 = 0$, $V_2 = w_G$, $V_3 = w_L$, $V_4 = w_L$, $V_5 = w_L$,

$$V_6 = 2w_{Gc} - \frac{\mu_p^{t}}{\alpha_G \rho'_{G/p,s}}.$$

They have the same sign as $w_G$ and $w_L$, provided the restriction [43] is satisfied.

### 4.2.3. *Discussion*

As found for single-phase flows, the presence of differential terms in the expressions of the interfacial transfer laws is fully compatible with the interpretation of the critical phenomenon in terms of propagation of small disturbances. To the restriction on the values of the $\eta_i$ and $\zeta_i$ in single phase, correspond here restrictions on the values of the $\lambda_i$, $\mu_i$, $\nu_i$.

## 5. CONCLUSIONS

5.1. To improve the understanding of two-phase critical flow phenomena, it has been found useful to study in parallel both single- and two-phase flows. It has also been pointed out that to postulate the transfer laws, which are the causes of the evolution of the fluid, is more rational than to assume *a priori* the nature of this evolution. This point is especially important for two-phase flows because of the presence of interfacial transfers, which are deemed to play an essential part in mixture evolution.

5.2. In the "classical" analysis, the transfer terms present in the equations are assumed to be only functions of space, time, and of the dependent variables used to describe the flow. Within this frame, the results found for critical flow rates and propagation velocities of small disturbances are always the same (when all the conservation equations are used, which is, of course, imperative):

For single-phase flows, the critical velocity is the isentropic sound velocity, which is generally in agreement with the experimental data.

For two-phase flows, the calculated critical flow rate [34], is *not* in agreement with the experimental data. Moreover, unlike what may be expected, it does not depend on the interfacial transfer phenomena.

5.3. A consequence of this uniqueness is the incompatibility of the various existing two-phase models, which give different results: it is *not* possible, starting from a general model and making appropriate assumptions, to arrive at these existing models.

5.4. To get rid of the foregoing drawbacks, it is proposed here to adopt a more general form for the transfer laws, i.e. to allow for the presence in their expressions of partial derivatives of the dependent variables. The examples dealt with in the paper show that this is a promising hypothesis: results other than those recalled above (5.2) can be obtained, such as a "polytropic" critical velocity for single phase flow or the results of existing models for two-phase flow. The importance of the foregoing differential terms proceeds from the fact that they strongly affect the coupling between the phases (or between the fluid and the wall).

5.5. The critical flow condition has been mathematically studied: a necessary critical flow criterion is obtained by equating to zero the determinant of the set of equations describing the steady state flow [14].

This criterion must be complemented by the study of the compatibility condition of the set [15]. It is to be emphasized that the gradients of the dependent variables are generally not infinite in the critical section.

5.6. It has been verified that a flow is critical, with two phases as well as with a single phase, when disturbances initiated downstream of some "critical" section cannot propagate upstream of this section. Thus a decrease of the outlet pressure, for example, has no effect on the flow parameters upstream of the critical section. As a result, the flow rate cannot be increased further. The presence of differential terms in the expressions of the transfer laws is fully compatible with this interpretation.

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